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Teaching about Fractions: What, When, and How?

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Learning about fractions is one of the most difficult tasks for middle and junior high school children. The results of the third National Assessment of Educational Progress (NAEP) show an apparent lack of understanding of fractions by nine-, thirteen-, and seventeen-year-olds. "The performance on fractional computation was low, and students seem to have done their computation with little understanding" (Lindquist et al. 1983, p. 16). Similar trends were observed in the first, the second, and the recently completed fourth National Assessments (Carpenter et al. 1978; 1980; Post 1981; Dossey et al. 1988). Even though operations on fractions are taught as early as grade 4, the second NAEP showed that only 35 percent of the thirteen- year-olds could correctly answer the test item 3/4 + 1/2.

The difficulty children have with fractions should not be surprising considering the complexity of the concepts involved. Children must adopt new rules for fractions that often conflict with well-established ideas about whole numbers. For example, when ordering fractions with like numerators, children learn that 1/3 is less than 1/2. With whole numbers, however, 3 is greater than 2. When comparing fractions of this type, children need to coordinate the inverse relationship between the size of the denominator and the size of the fraction. They need to realize that if a pie is divided into three equal parts, each piece will be smaller than when a pie of the same size is divided into two equal parts. With fractions, the *more* pieces, the *smaller* the size of each piece.

Another difficulty is that the rules for ordering fractions with like numerators do not apply to fractions with like denominators. In this situation, children can use their already learned ideas about counting. For example, a student might reason that the fraction 5/7 is greater than 2/7 because 5 is greater than 2.

When adding or subtracting fractions, children may have ideas about whole numbers that conflict with their ideas about fractions. An estimation item from the second NAEP reveals this phenomenon. Students were asked to estimate the answer to 12/13 + 7/8. The choices were 1, 2, 19, 21, and, "I don't know." Only 24 percent of the thirteen-year-olds responding chose the correct answer, 2. Fifty-five percent of the thirteen-year-olds selected 19 or 21-they added either the numerators or the denominators. These students seem to be operating on the fractions without any mental referents to aid their reasoning.

Clearly, the way fractions are taught must be improved. Because of the complexity of fraction concepts, more time should be allocated in the curriculum for developing students' understanding of fractions. But just more time is not sufficient to improve understanding; the emphasis of instruction should also shift from the development of algorithms for performing operations on fractions to the development of a quantitative understanding of fractions. For example, instruction should enable children to reason that the sum of 12/13 and 7/8 is about 2 because 12/13 is almost 1 and 7/8 is almost 1. Children should be able to reason that 1/2 + 1/3 cannot equal 2/ 5 because 2/5 is smaller than 1/2 and the sum must be bigger than either addend. Further, they should realize that 3/7 is less than 5/9 not because of a rule but because they know that 3/7 is less than 1/2 and 5/9 is greater than 1/2. Students who are able to reason in this way have a quantitative understanding of fractions.

To think quantitatively about fractions, students should know something about the relative size of fractions. They should be able to order fractions with the same denominators or same numerators as well as to judge if a fraction is greater than or less than 1/2. They should know the equivalents of 1/2 and other familiar fractions. The acquisition of a quantitative understanding of fractions is based on students' experiences with physical models and on instruction that emphasizes meaning rather than procedures.

This chapter presents suggestions for changes in the content and pace of instruction to help children develop both a quantitative understanding of, and skill in operating with, fractions. Specific suggestions are presented for what, when, and how fractions should be taught.

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RECOMMENDATIONS FOR CHANGE

Many current textbooks introduce fraction concepts as early as grade 2, though the main work on them begins in grade 4. The scope and sequence of the topic in grades 4, 5, and 6 are remarkably similar. The naming of fractions and the ideas of order and equivalence are briefly presented at each grade level; the most instructional time is allocated to operations with fractions. Addition and subtraction of fractions with like and unlike denominators are initially taught in grade 4 and repeated in grades 5 and 6. Multiplication and division of fractions are introduced in grade 5 and repeated in grade 6.

The result of this repetitious scope and sequence is that none of the topics is taught well. We suggest that by postponing most operations with fractions at the symbolic level until grade 6 and using instructional time in grades 4 and 5 to develop fraction concepts and the ideas of order and equivalence, teachers will find that their students will be more successful with all aspects of operations with fractions and will have a stronger quantitative under- standing of them.

We shall present our recommendations in two ways: (a) general recommendations applicable to instruction at all grade levels and (b) specific changes for the primary and intermediate grades.

General Recommendations

1. The use of manipulatives is crucial in developing students' understanding of fraction ideas. Manipulatives help students construct mental referents that enable them to perform fraction tasks meaningfully. Therefore, manipulatives should be used at each grade level to introduce all components of the curriculum on fractions. Manipulatives can include these models: fractional parts of circles, Cuisenaire rods, paper-folding activities, and counters (see fig. 13.1).

2. The proper development of concepts and relationships among fractions is essential if students are to perform and understand operations on fractions. The majority of instructional time before grade 6 should be devoted to developing these important notions.

3. Operations on fractions should be delayed until concepts and the ideas of the order and equivalence of fractions are firmly

established. Delaying work with operations will allow the necessary time for work on concepts.

4. The size of denominators used in computational exercises should be limited to the numbers 12 and below.

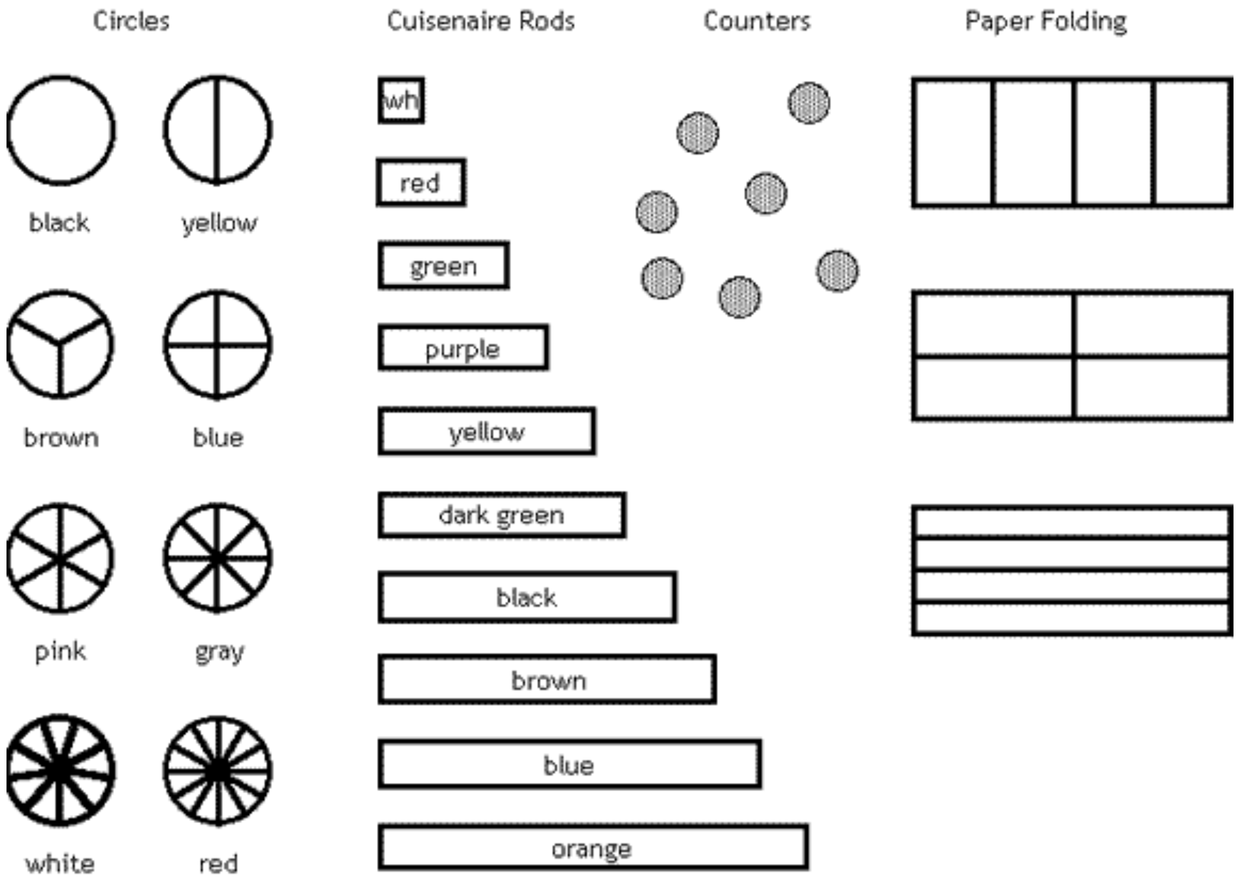

Fig. 13.1 Fraction manipulatives

Specific Recommendations

Primary grades. The major purpose of initial fraction-concept activities is to offer students experiences that will allow them to develop strong mental images of fractions. These images are the basis for quantitative understanding. Specific suggestions for early experiences with manipulatives in the primary grades are listed below.

1. Instruction should be based on the part-whole concept using first the continuous model (circles, paper folding) and then the discrete model (counters). The discrete model should be introduced by relating it to the circles.

2. Include activities that ask students to name fractions represented by physical models and diagrams. Unit and nonunit fractions with denominators no larger than 8 should be used. Also include activities

that ask students to model or draw pictures for fraction names or symbols.

3. Use words (three-fourths) initially and then introduce symbols (3/4).

4. Introduce "concept of unit" activities, that is, activities in which students name fractions when the unit is varied. For example, with the fraction circles in figure 13.1, state that each yellow piece is 1, rather than having the whole circle as the unit. Then ask students the value of the other pieces.

Intermediate grades. If the abstraction of fraction operations proceeds too quickly, the result may be a rote application of rules. Algorithms for the addition and subtraction of fractions with unlike denominators and the multiplication and division of fractions should be postponed until grade 6. Instruction in grades 4 and 5 should extend students' concepts of fractions, establish intuitive ordering strategies, and develop the idea of equivalent fractions. More specific suggestions for experiences with fractions in the intermediate grades are listed below.

1. Fraction-concept activities from the primary grades can include fractions with denominators no larger than 12 and problem-solving activities that extend students' concept of unit. For example, if a blue circular piece is called 1/3, then students can be asked to find the value of other circular pieces.

2. Fraction concepts can also be extended to new physical models (number lines or Cuisenaire rods) and to a new interpretation (e.g., the quotient model-3 pizzas shared by 4 people).

3. Activities for generating equivalent fractions should be introduced with manipulatives, then with diagrams. In particular, equivalent forms of such common fractions as 1/2, 1/3, 1/4, and 3/4 should be stressed, with the greatest emphasis placed on 1/2.

4. Comparisons of fractions with like denominators (2/7 and 3/7) and like numerators (2/4 and 2/8) should be developed with manipulatives. Children should be able to verbalize a rule for ordering fractions with like numerators that does not rely on changing them to equivalent fractions with like denominators.

5. Ordering pairs of fractions by comparing them to 1/2 or 1 should be included. For example, 3/ 10 is less than 2/3 because 3/10 is less than 1/2 and 2/3 is greater than 1/2. (Notice that although this pair of fractions has unlike numerators and unlike denominators, an ordering decision can be made without finding equivalent fractions with the same denominator).

6. The initial goal of instruction for addition and subtraction of fractions should be to model these operations with manipulatives and diagrams. Instruction should emphasize estimation and judging the reasonableness of answers.

7. Addition and subtraction of fractions at the symbolic level is appropriate in grade 6. Multiplication and division of fractions can also be introduced, but the goal should not be to calculate the answer, since it can easily be obtained using whole number ideas, with no conceptual understanding. Children should demonstrate an understanding of multiplication and division by modeling a problem with manipulatives or by naming the problem for the manipulative model. Students should be able to create story problems for a multiplication or division sentence or write a multiplication or division sentence for a story problem.

Specific teaching activities for some of the suggested curricular changes conclude this chapter. The activities are in three categories: (a) the concept of unit, (b) ordering, and (c) addition of fractions.

How to Teach Fractions

Concepts and ordering. As previously stated, an understanding of fraction concepts and order and equivalence relations is a prerequisite for success in computation with fractions. Behr, Lesh, Post, and Silver (1983) recommend using a variety of manipulatives to develop fraction concepts. The use of more than one manipulative enhances students' understanding and pro- motes abstraction of the concept from irrelevant perceptual features of a manipulative, such as color, size, or shape.

The following topics should be included in the teaching of fraction concepts, order, and equivalence: *(a)* modeling fractional amounts with more than one manipulative and naming unit and nonunit fractions; (*b*) generating equivalent fractions; (*c*) performing concept-of-unit activities; *(d)* ordering fractions. The following section describes sample activities for the last two topics. Post and Cramer (1987) and Bezuk (1988) present examples of activities for modeling fractions and generating equivalent fractions.

Concept-of-Unit Activities

Concept-of-unit activities require students to name fractional parts when the unit is varied. These activities strengthen their understanding of fraction concepts and reinforce mental images created in introductory activities with manipulatives. Figure 13.2 shows examples of activities with circles, counters, and Cuisenaire rods; in each case students are asked to name the fractions as the whole unit varies.

Teachers can extend concept-of-unit activities by including more difficult questions in which children are asked to reconstruct the whole given a unit or nonunit fraction. For example, if the blue circular piece is 1/2, what is the value of one pink piece? Or, if six counters equal 3/5, what is the value of two counters?

These questions reinforce the idea that, for example, two halves equal one whole, three thirds equal one whole, and so on. They also reinforce the notion that nonunit fractions are iterations of unit fractions (3/5 = 1/5 + 1/5 + 1/5).

A. Use fraction circles to solve these problems:

The black circle is 1. What is the value of each of these pieces?

1 blue	3 grays	1 pink	3 yellows

(Change the unit: The yellow piece is 1. Now what is the value of those pieces?)

B. Use counters to solve these problems:

Eight counters equal 1. What is the value of each of these sets of counters?

1 counter	2 counters	4 counters	6 counters

(Change the unit: Four counters equal 1. Now what is the value of those sets of counters?)

C. Use Cuisenaire rods to solve these problems:

The green Cuisenaire rod equals 1. What is the value of each of these rods?

red	black	white	dark green

(Change the unit: The dark green rod is 1. Now what is the value of those rods?)

Fig. 13.2 Concept-of-unit Activities

Ordering

Many pairs of fractions can be compared without using a formal algorithm, such as finding a common denominator or changing each fraction to a decimal. Children need informal ordering schemes to estimate fractions quickly or to judge the reasonableness of answers. They can be led to discover these relationships if they have had experiences in constructing mental images of fractions. Figure 13.3 shows four categories of fractions that can be compared without a formal algorithm.

1.	Pairs of fractions with like denominators:	
	1/4 and 3/4	3/5 and 4/5
2.	Pairs of fractions with like numerators:	
	1/3 and 1/2	2/5 and 2/3
3.	Pairs of fractions that are on opposite sides of 1/2 or 1:	
	3/7 and 5/9	3/11 and 11/3
4.	Pairs of fractions that have the same number of pieces less than one whole:	
	2/3 and 3/4	3/5 and 6/8

Fig. 13.3 Comparing fractions with mental images

The first two types of comparison problems are fairly straightforward when developed with manipulatives. Children have more difficulty with the second category because of the inverse relationship between the number of pieces the whole is divided into and the size of the denominator. Manipulatives are crucial in the development of an understanding of this relationship.

The last two categories of ordering problems were described to the authors by fourth-grade students who had used manipulatives to learn fraction concepts for several weeks. These strategies were not taught to the students and seemed to be self-generated. The fact that the students could create these two ordering procedures clearly demonstrates the power of using manipulatives to develop quantitative understanding of fractions.

The fractions in the third category are on "opposite sides" of a comparison point. Students order such fractions by comparing each to a known fraction, such as 1/2, and combining these results to make a final decision. One fourth-grade student compared 3/7 and 5/9 in the following manner (Roberts 1985): "Three-sevenths is less. It doesn't cover half the unit. Five-ninths covers over half." Notice that the child's reasoning is based on an internal image constructed for fractions.

The last type of comparison problem concerns pairs of fractions for which the difference between the numerator and denominator is the same. These pairs can be compared by determining how much less than a whole each fraction is. The remaining portion of the whole will represent fractions with the same numerator. Students then compare these fractions and reverse their decision for the answer. A fourth-grade student compared 6/8 and 3/5 in this way (Roberts 1985): "Six-eighths is greater. When you look at it, then you have six of them, and there'd be only two pieces left. And then if they're smaller pieces like, it wouldn't have very much space left in it, and it would cover up a lot more. Now here [3/5] the pieces are bigger, and if you have three of them you would still have two big ones left. So it would be less."

All four types of comparison problems can readily be solved with mental images developed by using manipulatives. Figures 13.4 and 13.5 present ordering activities by which teachers can lead students to discover rules for ordering fractions.

Exploring fractions with the same denominators

Use circular pieces. The whole circle is the unit.

A. Show 1/4 B. Show 3/4

Are the pieces the same size?

How many pieces did you use to show 1/4?

How many pieces did you us e to show 3/4?

Which fraction is larger?

Exploring fractions with the same numerator

Use your circular pieces. The whole circle is the unit. Take out the pieces listed in each problem; then answer the questions.

Pieces	How many cover 1 whole circle?	Which color takes more pieces to cover 1 whole?	Which color has the smaller pieces?
1. Brown	3		
Pink	6	x	x
2. Blue			
Yellow			

Fig 13.4 Activities for ordering fractions

Comparing fractions to 1/2 or 1

Use circular pieces. The whole circle is the unit.

A. Show 2/3 B. Show 1/4

Which fraction covers more than one-half of the circle?

Which fraction covers less than one-half of the circle?

Which fraction is larger?

Compare these fraction pairs in the same way.

2/8 and 3/5 1/3 and 5/6 32 and 2/3

Comparing fractions by deciding how close each fraction is to 1

Use your circular pieces. The whole circuit is the unit. Take out the pieces listed in each problem, then answer the questions.

Pieces	How much more is needed to make 1 whole?	Which piece is bigger?	Which display shows more?
2/3	1/3	x	
3/4	1/4		x
4/5			
6/8			

Fig 13.5 More activities for ordering fractions

Addition of Fractions

The initial goal for instruction on operations with fractions should be to have pupils add and subtract fractions using a manipulative. Fraction circles are excellent materials for modeling the addition and subtraction of fractions. An addition problem with like denominators, 1/6 + 4/6, for example, can be demonstrated like this: One pink piece (1/6) can be placed on the unit circle, followed by four pink pieces (4/6). The sum is represented by the fraction of the whole circle that is covered.

To add fractions with unlike denominators, children should understand why both fractions are converted to those with like denominators. To demonstrate this idea, first model sums of fractions with unlike denominators using fractions that have equivalences that students already know. For example, model 1/2 + 1/4 by placing a yellow piece and blue piece on a unit circle. Children can easily see that three-fourths of the circle is covered. To discuss the need for a common denominator, ask such questions as "How is this problem different from adding fractions like 1/8 + 5/8?" "How can you know for sure that 3/4 is covered if the circle has two different colored pieces on it?" "Can you show an equivalent problem using pieces of the same color?"

Whether they are adding fractions with like or unlike denominators, it is important to have students estimate the size of the sum before they model the problem with manipulatives. Such questions as "Is the answer greater than 1/2 or less than 1/2?" "Greater than 1 or less than 1?" force children to think quantitatively about fractions. To answer these questions, students can draw on their earlier use of manipulatives in naming and ordering fractions.

Reflecting on a frequently given incorrect sum is also important. For example, by asking if the sum of 1/6 and 4/6 could be 5/12, which is less than 1/2, the teacher can make students aware of the unreasonableness of adding numerators and denominators. With the concrete experiences recommended here, a student should be able to respond that since 5/12 is less than 1/2, it cannot be the correct answer. This observation is correct, since one of the addends (4/6) is greater than 1/2. Notice how a student giving this response uses the ordering strategies developed previously.

The next step in finding the sums of fractions with unlike denominators is to use pairs of fractions whose sums are more difficult to see from the model, for example, 1/3 + 3/6 or 1/3 + 1/4. The steps for modeling the sum of 1/3 and 1/4 with circular pieces are described below:

Estimate the answer first: " Is the answer greater than 1/2? Less than 1/2? Greater than 1? Less than 1?" Now demonstrate the problem with the circular pieces: place the blue piece and the brown piece in the unit circle and reflect on the accuracy of the estimates. Now ask, "Can you show 1/4 as an equivalent fraction using the brown pieces? Can you show 1/3 as an equivalent fraction with the blue pieces?" Since neither fraction can be shown with the other color, set aside time for students to use the circular pieces to find all the fractions equal to 1/4 and all those equal to 1/3 until they find one color that they can use to show both 1/4 and 1/3. Figure 13.6 illustrates this procedure.

By laying the pieces side by side, children can see that both 1/4 and 1/3 can be shown with the red pieces. One blue (1/4) equals three reds (3/12); one brown (1/3) equals four reds (4/12). Exchanges can be made with the circular pieces and the sum found.

Students should also reflect on why 2/7 is an unreasonable answer to this problem. They should be given opportunities to solve many sums with the circular pieces. When students are ready for work at the symbolic level, teachers can help them see the link between their physical actions with the fraction circles and the algorithm they will be taught.

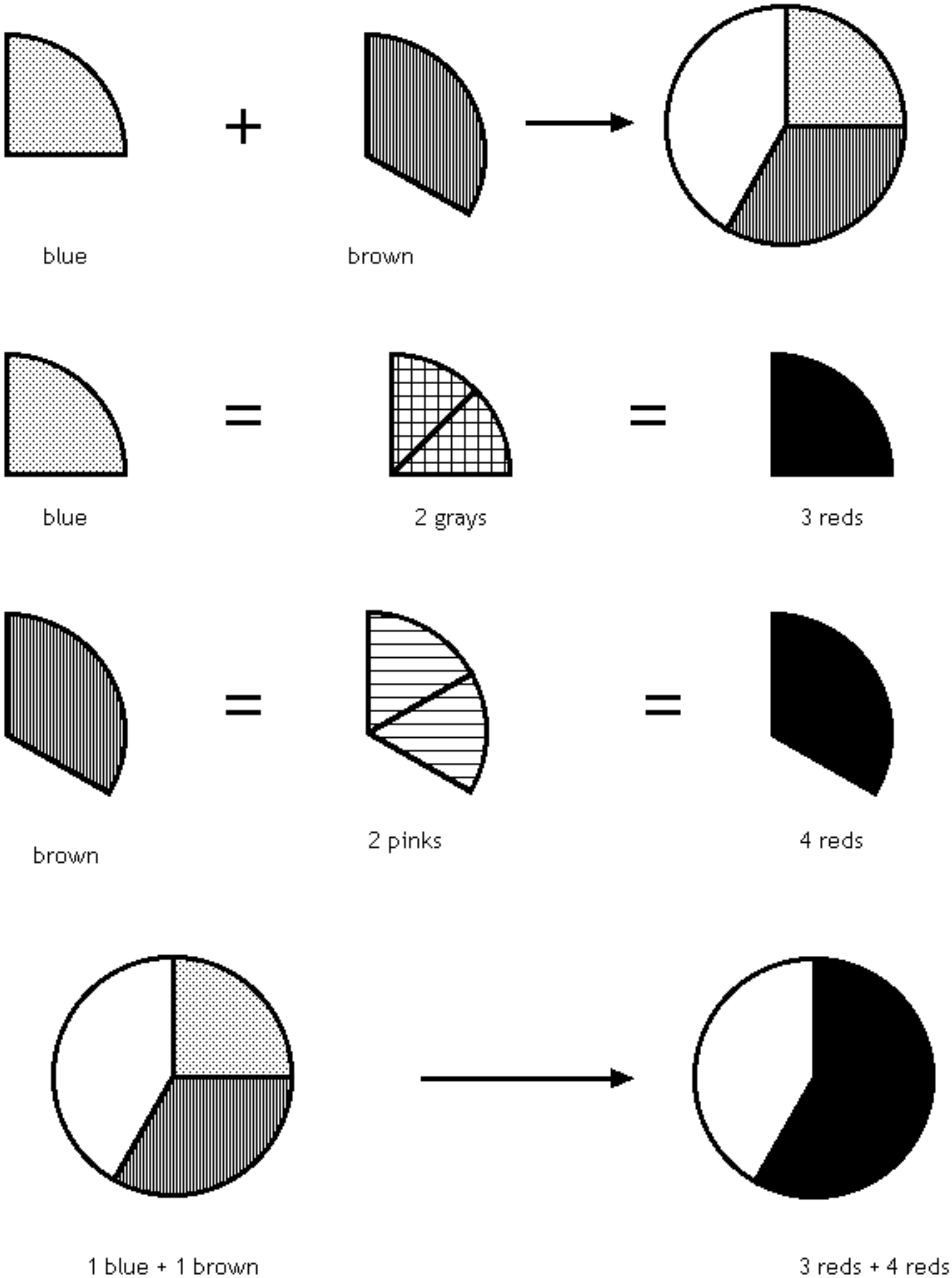

fig 13.6 Using fraction circles tomodel 1/3 + 1/4

CONCLUSION

We encourage teachers and schools to implement more appropriate objectives for teaching fractions in elementary school. We encourage teachers to use an instructional approach that emphasizes student involvement, the use of manipulatives, and the development of understanding before beginning work with formal symbols and operations.

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